

ASYMPTOTIC OF THE DISTRIBUTION AND HARMONIC MOMENTS FOR A SUPERCRITICAL BRANCHING PROCESS IN A RANDOM ENVIRONMENT

ION GRAMA, QUANSHENG LIU, AND ERIC MIQUEU

ABSTRACT. Let (Z_n) be a supercritical branching process in an independent and identically distributed random environment ξ . We show the exact decay rate of the probability $\mathbb{P}(Z_n = j | Z_0 = k)$ as $n \rightarrow \infty$, for each $j \geq k$, assuming that $\mathbb{P}(Z_1 = 0) = 0$. We also determine the critical value for the existence of harmonic moments of the random variable $W = \lim_{n \rightarrow \infty} \frac{Z_n}{\mathbb{E}(Z_n | \xi)}$ under a simple moment condition.

RÉSUMÉ. Soit (Z_n) un processus de branchement surcritique en environnement aléatoire ξ indépendant et identiquement distribué. Nous donnons un équivalent de la probabilité $\mathbb{P}(Z_n = j | Z_0 = k)$ lorsque $n \rightarrow \infty$, pour tout $j \geq k$, sous la condition $\mathbb{P}(Z_1 = 0) = 0$. Nous déterminons également la valeur critique pour l'existence des moments harmoniques de la variable aléatoire limite $W = \lim_{n \rightarrow \infty} \frac{Z_n}{\mathbb{E}(Z_n | \xi)}$, sous une hypothèse simple d'existence de moments.

1. INTRODUCTION

A branching process in a random environment (BPRE) is a natural and important generalisation of the Galton-Watson process, where the reproduction law varies according to a random environment indexed by time. It was introduced for the first time in Smith and Wilkinson [17] to modelize the growth of a population submitted to an environment. For background concepts and basic results concerning a BPRE we refer to Athreya and Karlin [4, 3]. In the critical and subcritical regime the branching process goes out and the research interest is mostly concentrated on the survival probability and conditional limit theorems, see e.g. Afanasyev, Böinghoff, Kersting, Vatutin [1, 2], Vatutin [19], Vatutin and Zheng [20], and the references therein. In the supercritical case, a great deal of current research has been focused on large deviations, see Bansaye and Berestycki [6], Bansaye and Böinghoff [7, 8, 9], Böinghoff and Kersting [11], Huang and Liu [13], Nakashima [16]. In the particular case when the offspring distribution is geometric, precise asymptotics can be found in Böinghoff [10], Kozlov [14].

Date: June 15, 2016.

2010 Mathematics Subject Classification. Primary 60J80, 60K37, 60J05. Secondary 60J85, 92D25.

Key words and phrases. Branching processes, random environment, harmonic moments, asymptotic distribution, decay rate.

An important closely linked issue is the asymptotic behavior of the distribution of a BPRE (Z_n) , i.e. the limit of $\mathbb{P}(Z_n = j | Z_0 = k)$ as $n \rightarrow \infty$, for fixed $j \geq 1$ when the process starts with $k \geq 1$ initial individuals. For the Galton-Watson process, the asymptotic is well-known and can be found in the book by Athreya [5]. For the need of the lower large deviation principle of a BPRE, Bansaye and Böinghoff have shown in [9] that, for any fixed $j \geq 1$ and $k \geq 1$ it holds $n^{-1} \log \mathbb{P}(Z_n = j | Z_0 = k) \rightarrow -\rho$ as $n \rightarrow \infty$, where $\rho > 0$ is a constant. This result characterizes the exponential decrease of the probability $\mathbb{P}(Z_n = j | Z_0 = k)$ for the general supercritical case, when extinction can occur. However, it stands only on a logarithmic scale, and the constant ρ is not explicit, except when the reproduction law is fractional linear, for which ρ is explicitly computed in [9]. Sharper asymptotic results for the fractional linear case can be found in [10]. In the present paper, we improve the results of [9] and extend those of [10] by giving an equivalent of the probability $\mathbb{P}(Z_n = j | Z_0 = k)$ as $n \rightarrow \infty$, provided that each individual gives birth to at least one child. These results are important to understand the asymptotic law of the process, and are useful to obtain sharper asymptotic large deviation results. We also improve the result of [13] about the critical value for the harmonic moment of the limit variable $W = \lim_{n \rightarrow \infty} \frac{Z_n}{\mathbb{E}(Z_n | \xi)}$.

Let us explain briefly the findings of the paper. Assume that $\mathbb{P}(Z_1 = 0) = 0$. From Theorem 2.3 of the paper it follows that when $Z_0 = 1$,

$$(1.1) \quad \mathbb{P}(Z_n = j) \underset{n \rightarrow \infty}{\sim} \gamma^n q_j \quad \text{with} \quad \gamma = \mathbb{P}(Z_1 = 1) > 0,$$

where $q_j \in (0, +\infty)$ can be computed as the unique solution of some recurrence equations; moreover, the generating function $Q(t) = \sum_{j=1}^{\infty} q_j t^j$ has the radius of convergence equal to 1 and is characterized by the functional equation

$$(1.2) \quad \gamma Q(t) = \mathbb{E}Q(f_0(t)), \quad t \in [0, 1),$$

where $f_0(t) = \sum_{i=1}^{\infty} p_i(\xi_0) t^i$ is the conditional generating function of Z_1 given the environment. These results extend the corresponding results for the Galton-Watson process (see [5]). They also improve and complete the results in [9] and [10]: it was proved in [9] that $\frac{1}{n} \log \mathbb{P}(Z_n = j) \rightarrow \log \gamma$, and in [10] that $\mathbb{P}(Z_n = 1) \underset{n \rightarrow \infty}{\sim} \gamma^n q_1$ in the fractional linear case.

In the proofs of the above results we make use of Theorem 2.1 which shows that, with $m_0 = \mathbb{E}_{\xi} Z_1$, we have, for any fixed $a > 0$,

$$(1.3) \quad \mathbb{E}W^{-a} < \infty \quad \text{if and only if} \quad \mathbb{E}[p_1(\xi_0)m_0^a] < 1,$$

under a simple moment condition $\mathbb{E}[m_0^p] < \infty$ for some $p > a$, which is much weaker than the boundedness condition used in [13, Theorem 1.4] (see (2.10) below).

For the proof of Theorem 2.1 our argument consists of two steps. In the first step we prove the existence of the harmonic moment of some order $a > 0$ using the functional relation (3.5). The key argument to approach the critical value is in the second step, which is based on the method developed in [15, Lemma 4.1] obtaining the decay rate of the Laplace transform $\phi(t) = \mathbb{E}e^{-tW}$ as $t \rightarrow \infty$, starting from a

functional inequation of the form

$$(1.4) \quad \phi(t) \leq q\mathbb{E}\phi(Yt) + Ct^{-a},$$

where Y is a positive random variable. To prove (1.4) we use a recursive procedure for branching processes starting with k individuals and choosing k large enough. The intuition behind this consideration is that as the number of starting individuals k becomes larger, the decay rate of $\phi_k(t) = \mathbb{E}[e^{-tW}|Z_0 = k]$ as $t \rightarrow \infty$ is higher which leads to the desired functional inequation.

In the proof of Theorem 2.3, the equivalence relation (1.1) and the recursive equations for the limit values (q_j) come from simple monotonicity arguments. The difficulty is to characterize the sequence (q_j) by its generating function Q . To this end, we first calculate the radius of convergence of Q by determining the asymptotic behavior of the normalized harmonic moments $\mathbb{E}Z_n^{-r}/\gamma^n$ as $n \rightarrow \infty$ for some $r > 0$ large enough and by using the fact that $\sum_{j=1}^{\infty} j^{-r}q_j = \lim_{n \rightarrow \infty} \mathbb{E}Z_n^{-r}/\gamma^n$. We then show that the functional equation (1.2) has a unique solution subject to an initial condition.

The rest of the paper is organized as follows. The main results, Theorems 2.1 and 2.3, are presented in Section 2. Their proofs are given in Sections 3 and 4.

2. MAIN RESULTS

A BPRES (Z_n) can be described as follows. The random environment is represented by a sequence $\xi = (\xi_0, \xi_1, \dots)$ of independent and identically distributed random variables (i.i.d. r.v.'s), whose realizations determine the probability generating functions

$$(2.1) \quad f_n(t) = f(\xi_n, t) = \sum_{i=0}^{\infty} p_i(\xi_n)t^i, \quad t \in [0, 1], \quad p_i(\xi_n) \geq 0, \quad \sum_{i=0}^{\infty} p_i(\xi_n) = 1.$$

The branching process $(Z_n)_{n \geq 0}$ is defined by the relations

$$(2.2) \quad Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} N_{n,i}, \quad \text{for } n \geq 0,$$

where $N_{n,i}$ is the number of children of the i -th individual of the generation n . Conditionally on the environment ξ , the r.v.'s $N_{n,i}$ ($i = 1, 2, \dots$) are independent of each other with common probability generating function f_n , and also independent of Z_n .

In the sequel we denote by \mathbb{P}_ξ the *quenched law*, i.e. the conditional probability when the environment ξ is given, and by τ the law of the environment ξ . Then $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$ is the total law of the process, called *annealed law*. The corresponding quenched and annealed expectations are denoted respectively by \mathbb{E}_ξ and \mathbb{E} . We also denote by \mathbb{P}_k and \mathbb{E}_k the corresponding probability and expectation starting with k individuals. For $n \in \mathbb{N}$, the probability generating function of Z_n is

$$(2.3) \quad G_n(t) = \mathbb{E}t^{Z_n} = \mathbb{E}[f_0 \circ \dots \circ f_{n-1}(t)] = \mathbb{E}[g_n(t)],$$

where $g_n(t) = f_0 \circ \dots \circ f_{n-1}(t)$ is the conditional probability generating function of Z_n when the environment ξ is given. It follows from (2.2) that the probability generating function $G_{k,n}$ of Z_n starting with k individuals is

$$(2.4) \quad G_{k,n}(t) = \mathbb{E}_k t^{Z_n} = \mathbb{E} \left[g_n^k(t) \right].$$

We also define, for $n \geq 0$,

$$m_n = m(\xi_n) = \sum_{i=0}^{\infty} i p_i(\xi_n) \quad \text{and} \quad \Pi_n = \mathbb{E}_\xi Z_n = m_0 \dots m_{n-1},$$

where m_n represents the average number of children of an individual of generation n when the environment ξ is given. Let

$$(2.5) \quad W_n = \frac{Z_n}{\Pi_n}, \quad n \geq 0,$$

be the normalized population size. It is well known that under \mathbb{P}_ξ , as well as under \mathbb{P} , the sequence $(W_n)_{n \geq 0}$ is a non-negative martingale with respect to the filtration

$$\mathcal{F}_n = \sigma(\xi, N_{j,i}, 0 \leq j \leq n-1, i = 1, 2, \dots),$$

where by convention $\mathcal{F}_0 = \sigma(\xi)$. Then the limit $W = \lim_{n \rightarrow \infty} W_n$ exists \mathbb{P} -a.s. and $\mathbb{E}W \leq 1$.

We shall assume that

$$\mu := \mathbb{E} \log m_0 \in (0, \infty),$$

which implies that the BPFE is supercritical and that

$$(2.6) \quad \gamma := \mathbb{P}(Z_1 = 1) \in [0, 1).$$

With the extra condition $\mathbb{E}|\log(1 - p_0(\xi_0))| < \infty$ (see [17]), the population size tends to infinity with positive probability. We also assume in the whole paper that each individual gives birth to at least one child, i.e.

$$(2.7) \quad p_0(\xi_0) = 0 \quad a.s.$$

Therefore, under the condition

$$(2.8) \quad \mathbb{E} \frac{Z_1}{m_0} \log^+ Z_1 < \infty,$$

the martingale (W_n) converges to W in $L^1(\mathbb{P})$ (see e.g. [18]) and

$$\mathbb{P}(W > 0) = \mathbb{P}(Z_n \rightarrow \infty) = 1.$$

Our first result concerns the harmonic moments of the r.v. W .

Theorem 2.1. *Assume that there exists a constant $p > 0$ such that $\mathbb{E}[m_0^p] < \infty$. Then for any $a \in (0, p)$,*

$$\mathbb{E}_k W^{-a} < \infty \quad \text{if and only if} \quad \mathbb{E} \left[p_1^k(\xi_0) m_0^a \right] < 1.$$

From Theorem 2.1 we get the following corollary.

Corollary 2.2. *Let $a_k > 0$ be the solution of the equation*

$$(2.9) \quad \mathbb{E}[p_1^k m_0^{a_k}] = 1.$$

Assume that $\mathbb{E}m_0^{a_k} < \infty$. Then,

$$\begin{cases} \mathbb{E}_k W^{-a} < \infty & \text{for } a \in [0, a_k), \\ \mathbb{E}_k W^{-a} = \infty & \text{for } a \in [a_k, \infty). \end{cases}$$

The solution a_k of the equation (2.9) is the critical value for the existence of harmonic moments of the r.v. W . Note that, when the process starts with one individual, the critical value a_1 for the harmonic moments of W has been found in Theorem 1.4 of [13] under the more restrictive condition

$$(2.10) \quad A_1 \leq m_0 \quad \text{and} \quad \sum_{i=1}^{\infty} i^{1+\delta} p_i(\xi_0) \leq A^{1+\delta} \quad a.s.,$$

where $\delta > 0$ and $1 < A_1 < A$ are some constants. Theorem 2.1 and Corollary 2.2 generalize the result of [13], in the sense that we consider k initial individuals rather than just one and that the boundedness condition (2.10) is relaxed to the simple moment condition $\mathbb{E}[m_0^p] < \infty$.

The next result gives an equivalent as $n \rightarrow \infty$ of the probability $\mathbb{P}_k(Z_n = j) = \mathbb{P}(Z_n = j | Z_0 = k)$, with $k \in \mathbb{N}^*$ and $j \geq k$, in the case when $\mathbb{P}(Z_1 = 1) > 0$. The last condition implies that, for $k \geq 1$,

$$(2.11) \quad \gamma_k = \mathbb{P}_k(Z_1 = k) = \mathbb{E}[p_1^k(\xi_0)] > 0.$$

Define r_k as the solution of the equation

$$(2.12) \quad \gamma_k = \mathbb{E}m_0^{-r_k}.$$

Theorem 2.3. *Assume that $\mathbb{P}(Z_1 = 1) > 0$. For any $k \geq 1$ the following assertions holds.*

a) *For any accessible state $j \geq k$ in the sense that $\mathbb{P}_k(Z_l = j) > 0$ for some $l \geq 0$, we have*

$$(2.13) \quad \mathbb{P}_k(Z_n = j) \underset{n \rightarrow \infty}{\sim} \gamma_k^n q_{k,j},$$

where $q_{k,k} = 1$ and, for $j > k$, $q_{k,j} \in (0, +\infty)$ is the solution of the recurrence relation

$$(2.14) \quad \gamma_k q_{k,j} = \sum_{i=k}^j p(i, j) q_{k,i},$$

with $q_{k,i} = 0$ for any non-accessible state i , i.e. $\mathbb{P}_k(Z_l = i) = 0$ for all $l \geq 0$.

b) *Assume that there exists $\varepsilon > 0$ such that $\mathbb{E}[m_0^{r_k + \varepsilon}] < \infty$. Then, for any $r > r_k$, we have $\sum_{j=k}^{\infty} j^{-r} q_{k,j} < \infty$. In particular the radius of convergence of the power series*

$$(2.15) \quad Q_k(t) = \sum_{j=k}^{+\infty} q_{k,j} t^j$$

is equal to 1.

c) For all $t \in [0, 1)$ and $k \geq 1$, we have,

$$(2.16) \quad \frac{G_{k,n}(t)}{\gamma_k^n} \uparrow Q_k(t) \quad \text{as } n \rightarrow \infty,$$

where $G_{k,n}$ is the probability generating function of Z_n when $Z_0 = k$, defined in (2.4).

d) $Q_k(t)$ is the unique power series which verifies the functional equation

$$(2.17) \quad \gamma_k Q_k(t) = \mathbb{E}[Q_k(f_0(t))], \quad t \in [0, 1),$$

with the condition $Q_k^{(k)}(0) = 1$.

Part a) improves the bound $\mathbb{P}(Z_n \leq j) \leq n^j \gamma^n$ obtained in [6] (Lemma 7) for a BPPE with $\mathbb{P}(Z_1 = 0) = 0$. Furthermore, Theorem 2.3 extends the results of [5] for the Galton-Watson process, with some significant differences. Indeed, when the environment is random and non-degenerate, we have, for $k \geq 2$, $G_{k,1}(t) = \mathbb{E}f_0^k(t) \neq G_1^k(t)$ in general, which implies that $Q_k(t) \neq Q^k(t)$, whereas we have the relation $Q_k(t) = Q^k(t)$ for the Galton-Watson process.

Theorem 2.3 also improves the results of [9] (Theorem 2.1), where it has been proved that for a general supercritical BPPE

$$(2.18) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_k(Z_n = j) = -\rho < 0.$$

Our result is sharper in the case where $\mathbb{P}(Z_1 = 0) = 0$. Moreover, in the case where $\mathbb{P}(Z_1 = 0) = 0$, it has been stated mistakenly in [9] that $\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_k(Z_n = j) = k \log \gamma$, whereas the correct asymptotic is

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_k(Z_n = j) = \log \gamma_k.$$

Now we discuss the particular fractional linear case. The reproduction law of a BPPE is said to be fractional linear if

$$(2.19) \quad p_0(\xi_0) = a_0, \quad p_k(\xi_0) = \frac{(1 - a_0)(1 - b_0)}{b_0} b_0^k,$$

with generating function f_0 given by

$$f_0(t) = a_0 + \frac{(1 - a_0)(1 - b_0)t}{1 - b_0 t},$$

where $a_0 \in [0, 1)$, $b_0 \in (0, 1)$, with $a_0 + b_0 \leq 1$, are random variables depending on the environment ξ_0 . In this case, the mean of the offspring distribution is given by

$$m_0 = \frac{1 - a_0}{1 - b_0}.$$

The constant ρ in (2.18) was computed in [9]: with $X = \log m_0$,

$$\rho = \begin{cases} -\log \mathbb{E}[e^{-X}] & \text{if } \mathbb{E}[X e^{-X}] \geq 0 \quad (\text{intermediately and} \\ & \text{strongly supercritical case}), \\ -\log \inf_{\lambda \geq 0} \mathbb{E}[e^{-\lambda X}] & \text{if } \mathbb{E}[X e^{-X}] < 0 \quad (\text{weakly supercritical case}). \end{cases}$$

Moreover, precise asymptotic results for the strongly and intermediately supercritical case can be found in [10], where the following assertions are proved:

- (1) if $\mathbb{E}[Xe^{-X}] > 0$ (strongly supercritical case),

$$\mathbb{P}(Z_n = 1) \sim \nu \left(\mathbb{E}[e^{-X}] \right)^n ;$$

- (2) if $\mathbb{E}[Xe^{-X}] = 0$ (intermediately supercritical case),

$$\mathbb{P}(Z_n = 1) \sim \theta \left(\mathbb{E}[e^{-X}] \right)^n l(n) n^{-(1-s)},$$

with θ, ν, s positive constants and $l(\cdot)$ a slowly varying function. In the particular case where $a_0 = 0$, Theorem 2.3 recovers Theorem 2.1.1 of [10] with $p_1(\xi_0) = 1/m_0$, $X = \log m_0 > 0$ and $\mathbb{E}[Xe^{-X}] > 0$. Therefore the process is strongly supercritical and $\mathbb{P}(Z_n = 1) \sim \nu \left(\mathbb{E}[e^{-X}] \right)^n = \gamma^n$. However, since we assume $\mathbb{P}(Z_1 = 0) = 0$, our result does not highlight the previous two asymptotic regimes stated in the particular case when the distribution is fractional linear. The study of the general case is a challenging problem which still remains open.

3. HARMONIC MOMENTS OF W

In this section we prove Theorem 2.1. Denote the quenched Laplace transform of W under the environment ξ by

$$(3.1) \quad \phi_\xi(t) = \mathbb{E}_\xi \left[e^{-tW} \right],$$

and the annealed Laplace transform of W starting with k individuals by

$$(3.2) \quad \phi_k(t) = \mathbb{E}_k [\phi_\xi(t)] = \mathbb{E} [\phi_\xi^k(t)] = \mathbb{E}_k [e^{-tW}].$$

We start with a lemma which gives a lower bound for the harmonic moment of W .

Lemma 3.1. *Assume that $\mathbb{E}[m_0^p] < \infty$ for some constant $p > 0$. For any $k \geq 1$, let*

$$(3.3) \quad \alpha_k = \frac{p}{1 - \log \mathbb{E} m_0^p / \log \gamma_k},$$

with the convention that $\alpha_k = p$ if $p_1(\xi_0) = 0$ a.s. (so that $\gamma_k = 0$). Then, for all $a \in (0, \alpha_k)$,

$$\mathbb{E}_k W^{-a} < \infty.$$

Furthermore, if $\mathbb{P}(p_1(\xi_0) = 0) < 1$, we have $\alpha_k < \alpha_{k+1}$; if additionally $\mathbb{P}(p_1(\xi_0) < 1) = 1$, then $\lim_{k \rightarrow \infty} \alpha_k = p$.

Proof. We use the same approach as in [12] where the case $k = 1$ was treated. Since W is a positive random variable, it can be easily seen that, for $\alpha > 0$,

$$(3.4) \quad \mathbb{E}_k W^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \phi_k(t) t^{\alpha-1} dt,$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the Gamma function. Moreover, it is well-known that $\phi_\xi(t)$ satisfies the functional relation

$$(3.5) \quad \phi_\xi(t) = f_0 \left(\phi_{T\xi} \left(\frac{t}{m_0} \right) \right),$$

where $f_0(t) = \sum_{k=1}^{\infty} p_k(\xi_0)t^k$ is the generating function of Z_1 under ξ_0 , defined in (2.1). Using (3.5) and the fact that $\phi_{T\xi}^k\left(\frac{t}{m_0}\right) \leq \phi_{T\xi}^2\left(\frac{t}{m_0}\right)$ for all $k \geq 2$, we obtain

$$(3.6) \quad \phi_{\xi}(t) \leq p_1(\xi_0)\phi_{T\xi}\left(\frac{t}{m_0}\right) + (1 - p_1(\xi_0))\phi_{T\xi}^2\left(\frac{t}{m_0}\right).$$

Taking the k -th power in (3.6), using the binomial expansion and the fact that $\phi_{T\xi}^{2k-i}\left(\frac{t}{m_0}\right) \leq \phi_{T\xi}^{k+1}\left(\frac{t}{m_0}\right)$ for all $i \in \{0, \dots, k-1\}$, we get

$$(3.7) \quad \begin{aligned} \phi_{\xi}^k(t) &= p_1^k(\xi_0)\phi_{T\xi}^k\left(\frac{t}{m_0}\right) + \sum_{i=0}^{k-1} C_k^i p_1(\xi_0)^i (1 - p_1(\xi_0))^{k-i} \phi_{T\xi}^{2(k-i)+i}\left(\frac{t}{m_0}\right) \\ &\leq p_1^k(\xi_0)\phi_{T\xi}^k\left(\frac{t}{m_0}\right) + (1 - p_1^k(\xi_0))\phi_{T\xi}^{k+1}\left(\frac{t}{m_0}\right) \\ &= \phi_{T\xi}^k\left(\frac{t}{m_0}\right) \left[p_1^k(\xi_0) + (1 - p_1^k(\xi_0))\phi_{T\xi}\left(\frac{t}{m_0}\right) \right]. \end{aligned}$$

By iteration, this leads to

$$(3.8) \quad \phi_{\xi}^k(t) \leq \phi_{T^n\xi}^k\left(\frac{t}{\Pi_n}\right) \prod_{j=0}^{n-1} \left(p_1^k(\xi_j) + (1 - p_1^k(\xi_j))\phi_{T^n\xi}\left(\frac{t}{\Pi_n}\right) \right).$$

Taking expectation and using the fact that $\phi_{T^n\xi}(\cdot) \leq 1$, we have

$$\phi_k(t) \leq \mathbb{E} \left[\prod_{j=0}^{n-1} \left(p_1^k(\xi_j) + (1 - p_1^k(\xi_j))\phi_{T^n\xi}\left(\frac{t}{\Pi_n}\right) \right) \right].$$

Since $\phi_{\xi}(\cdot)$ is non-increasing, using a truncation, we get for all $A > 1$,

$$\phi_k(t) \leq \mathbb{E} \left[\prod_{j=0}^{n-1} \left(p_1^k(\xi_j) + (1 - p_1^k(\xi_j))\phi\left(\frac{t}{A^n}\right) \right) \right] + \mathbb{P}(\Pi_n \geq A^n).$$

As $T^n\xi$ is independent of $\sigma(\xi_0, \dots, \xi_{n-1})$, and the r.v.'s $p_1(\xi_i)$ ($i \geq 0$) are i.i.d., we obtain

$$\phi_k(t) \leq \left[\gamma_k + (1 - \gamma_k)\phi\left(\frac{t}{A^n}\right) \right]^n + \mathbb{P}(\Pi_n \geq A^n),$$

where $\gamma_k = \mathbb{E}p_1^k(\xi_0)$ is defined in (2.11). By the dominated convergence theorem, we have $\lim_{t \rightarrow \infty} \phi(t) = 0$. Thus, for any $\delta \in (0, 1)$, there exists a constant $K > 0$ such that, for all $t \geq K$, we have $\phi(t) \leq \delta$. Consequently, for all $t \geq KA^n$, we have $\phi\left(\frac{t}{A^n}\right) \leq \delta$ and

$$(3.9) \quad \phi_k(t) \leq \beta^n + \mathbb{P}(\Pi_n \geq A^n),$$

where

$$(3.10) \quad \beta = \gamma_k + (1 - \gamma_k)\delta \in (0, 1).$$

Using Markov's inequality, we have $\mathbb{P}(\Pi_n \geq A^n) \leq (\mathbb{E}m_0^p/A^p)^n$. Setting $A = \left(\frac{\mathbb{E}m_0^p}{\beta}\right)^{1/p} > 1$, we get for any $n \in \mathbb{N}$ and $t \geq KA^n$,

$$(3.11) \quad \phi_k(t) \leq 2\beta^n.$$

Now, for any $t \geq K$, define $n_0 = n_0(t) = \left\lfloor \frac{\log(t/K)}{\log A} \right\rfloor \geq 0$, where $[x]$ stands for the integer part of x , so that

$$\frac{\log(t/K)}{\log A} - 1 \leq n_0 \leq \frac{\log(t/K)}{\log A} \quad \text{and} \quad t \geq K A^{n_0}.$$

Then, for $t \geq K$,

$$\phi_k(t) \leq 2\beta^{n_0} \leq 2\beta^{-1}(t/K)^{\frac{\log \beta}{\log A}} = C_0 t^{-\alpha},$$

with $C_0 = 2\beta^{-1}K^\alpha$ and $\alpha = -\frac{\log \beta}{\log A} > 0$. Thus, we can choose a constant $C > 0$ large enough, such that, for all $t > 0$,

$$(3.12) \quad \phi_k(t) \leq C t^{-\alpha}.$$

Furthermore, by the definition of β , A and α , we have

$$\alpha = \frac{p}{1 - \log \mathbb{E} m_0^p / \log(\gamma_k + (1 - \gamma_k)\delta)},$$

where $\delta \in (0, 1)$ is an arbitrary constant and $\gamma_k = \mathbb{E} p_1^k(\xi_0)$. When $\delta \rightarrow 0$, we have $\alpha \rightarrow \alpha_k$, so that (3.12) holds for all $\alpha < \alpha_k$, where α_k is defined in (3.3). By (3.4) and (3.12), we conclude that $\mathbb{E} W^{-\alpha} < \infty$ for any $\alpha < \alpha_k$. Moreover, it is easily seen that if $\mathbb{P}(p_1(\xi_0) = 0) < 1$, then $\alpha_k < \alpha_{k+1}$ since $\gamma_{k+1} < \gamma_k$; if additionally $\mathbb{P}(p_1(\xi_0) < 1) = 1$, then $\lim_{k \rightarrow \infty} \gamma_k = 0$ so that $\lim_{k \rightarrow \infty} \alpha_k = p$. \square

The following lemma is the key technical tool to study the exact decay rate of the Laplace transform of the limit variable W .

Lemma 3.2 ([15], Lemma 4.1). *Let $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a bounded function and let Y be a positive random variable such that for some constants $q \in (0, 1)$, $a \in (0, \infty)$, $C > 0$ and $t_0 \geq 0$ and all $t > t_0$,*

$$\phi(t) \leq q \mathbb{E} \phi(Yt) + C t^{-a}.$$

If $q \mathbb{E}(Y^{-a}) < 1$, then $\phi(t) = O(t^{-a})$ as $t \rightarrow \infty$.

Now we proceed to prove Theorem 2.1. We first prove the necessity. Assume that $\mathbb{E}_k W^{-a} < \infty$ for some $a > 0$. We shall show that $\mathbb{E} p_1^k(\xi_0) m_0^a < 1$. Note that the r.v. W admits the well-known decomposition

$$W = \frac{1}{m_0} \sum_{i=1}^{Z_1} W(i),$$

where the r.v.'s $W(i)$ ($i \geq 1$) are i.i.d. and independent of Z_1 under \mathbb{P}_ξ , and are also independent of Z_1 and ξ_0 under \mathbb{P} . The conditional probability law of $W(i)$ satisfies $\mathbb{P}_\xi(W(i) \in \cdot) = \mathbb{P}_{T\xi}(W \in \cdot)$. Since $\mathbb{P}_k(Z_1 \geq k+1) > 0$, we have

$$(3.13) \quad \mathbb{E}_k W^{-a} > \mathbb{E}_k m_0^a \left(\sum_{i=1}^{Z_1} W(i) \right)^{-a} \mathbb{1}\{Z_1 = k\} = \mathbb{E} p_1^k(\xi_0) m_0^a \mathbb{E}_k W^{-a},$$

which implies that $\mathbb{E} p_1^k(\xi_0) m_0^a < 1$.

We now prove the sufficiency. Assume that $\mathbb{E} m_0^p < \infty$ and $\mathbb{E} p_1^k(\xi_0) m_0^a < 1$ for some $a \in (0, p)$.

We first consider the case where $\mathbb{P}(p_1(\xi_0) < 1) = 1$. We prove that $\mathbb{E}_k W^{-a} < \infty$ by showing that $\phi_k(t) = O(t^{-(a+\varepsilon)})$ as $t \rightarrow \infty$, for some $\varepsilon > 0$. By Lemma 3.1, there exists an integer $j \geq k$ large enough and a constant $C > 0$ such that

$$(3.14) \quad \phi_j(t) \leq C t^{-(a+\varepsilon)},$$

with $\varepsilon > 0$ and $a + \varepsilon < p$. By (3.7), we have

$$(3.15) \quad \phi_\xi^{j-1}(t) \leq p_1^{j-1}(\xi_0) \phi_{T\xi}^{j-1}\left(\frac{t}{m_0}\right) + \phi_{T\xi}^j\left(\frac{t}{m_0}\right).$$

Taking the expectation in (3.15), using (3.2), (3.14) and the independence between ξ_0 and $T\xi$, we obtain

$$(3.16) \quad \begin{aligned} \phi_{j-1}(t) &\leq \mathbb{E}\left[p_1^{j-1}(\xi_0) \phi_{j-1}\left(\frac{t}{m_0}\right)\right] + C t^{-(a+\varepsilon)} \\ &= \gamma_{j-1} \mathbb{E}[\phi_{j-1}(Yt)] + C t^{-(a+\varepsilon)}, \end{aligned}$$

where $\gamma_{j-1} = \mathbb{E}[p_1^{j-1}(\xi_0)] < 1$ and Y is a positive random variable whose distribution is determined by

$$\mathbb{E}[g(Y)] = \frac{1}{\gamma_{j-1}} \mathbb{E}\left[p_1^{j-1}(\xi_0) g\left(\frac{1}{m_0}\right)\right],$$

for all bounded and measurable function g . By hypothesis, $\mathbb{E}p_1^k(\xi_0)m_0^a < 1$. Then, by the dominated convergence theorem, there exists $\varepsilon > 0$ small enough such that $\mathbb{E}p_1^k(\xi_0)m_0^{a+\varepsilon} < 1$, and since $j-1 \geq k$, we have $\mathbb{E}p_1^{j-1}(\xi_0)m_0^{a+\varepsilon} \leq \mathbb{E}p_1^k(\xi_0)m_0^{a+\varepsilon} < 1$. Therefore, $\gamma_{j-1}\mathbb{E}[Y^{-(a+\varepsilon)}] < 1$ and using (3.16) and Lemma 3.2, we get $\phi_{j-1}(t) = O(t^{-(a+\varepsilon)})$ as $t \rightarrow \infty$. By induction, applying (3.15) and (3.16) to the functions $\phi_{j-2}, \phi_{j-3}, \dots, \phi_k$ and using the same argument as in the proof for ϕ_{j-1} , we obtain

$$(3.17) \quad \phi_k(t) = O(t^{-(a+\varepsilon)}) \quad \text{as } t \rightarrow \infty.$$

Therefore, in the case where $\mathbb{P}(p_1(\xi_0) < 1) = 1$, we have proved that

$$(3.18) \quad \mathbb{E}p_1^k(\xi_0)m_0^a < 1 \quad \text{implies} \quad \mathbb{E}_k W^{-a} < \infty.$$

Now consider the general case where $\mathbb{P}(p_1(\xi_0) < 1) < 1$. Denote the distribution of ξ_0 by τ_0 and define a new distribution $\tilde{\tau}_0$ as

$$(3.19) \quad \tilde{\tau}_0(\cdot) = \tau_0(\cdot | p_1(\xi_0) < 1).$$

Consider the new branching process whose environment distribution is $\tilde{\tau} = \tilde{\tau}_0^{\otimes \mathbb{N}}$ instead of $\tau = \tau_0^{\otimes \mathbb{N}}$. The corresponding probability and expectation are denoted by $\tilde{\mathbb{P}}(dx, d\xi) = \mathbb{P}_\xi(dx) \tilde{\tau}(d\xi)$ and $\tilde{\mathbb{E}}$, respectively. Of course (W_n) is still a martingale under $\tilde{\mathbb{P}}$. Moreover, the condition $\tilde{\mathbb{E}}\left[\frac{Z_1}{m_0} \log^+ Z_1\right] = \mathbb{E}\left[\frac{Z_1}{m_0} \log^+ Z_1\right] / \mathbb{P}(p_1(\xi_0) < 1) < \infty$ implies that $W_n \rightarrow W$ in $L^1(\tilde{\mathbb{P}})$. Now we show that $\mathbb{E}_k[W^{-a}] \leq \tilde{\mathbb{E}}_k[W^{-a}]$. For $0 \leq i \leq n$, denote

$$\begin{aligned} A_{i,n} &= \left\{ (\xi_0, \dots, \xi_{n-1}) \mid p_1(\xi_{j_1}) = \dots = p_1(\xi_{j_i}) = 1 \text{ for some } 0 \leq j_1 < \dots < j_i \leq n-1, \right. \\ &\quad \left. \text{and } p_1(\xi_h) < 1 \text{ for all } h \in \{0, \dots, n-1\} \setminus \{j_1, \dots, j_i\} \right\}. \end{aligned}$$

Conditioning by the events $A_{i,n}$ ($i \in \{0, \dots, n\}$) and using the fact that the r.v.'s ξ_0, \dots, ξ_{n-1} are i.i.d., we obtain, for all $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}_k [W_n^{-a}] &= \sum_{i=0}^n \mathbb{E}_k [W_n^{-a} | A_{i,n}] \mathbb{P}(A_{i,n}) \\ (3.20) \quad &= \sum_{i=0}^n \mathbb{E}_k [W_n^{-a} | A_{i,n}] C_n^i \eta^i (1 - \eta)^{n-i}, \end{aligned}$$

with $\eta = \mathbb{P}(p_1(\xi_0) = 1)$. Moreover, using (2.2), a straightforward computation leads to the decomposition

$$(3.21) \quad W_n = \prod_{i=0}^{n-1} \eta_i, \quad \text{with } n \geq 1 \quad \text{and} \quad \eta_i = \frac{1}{Z_i} \sum_{j=1}^{Z_i} \frac{N_{i,j}}{m_i}.$$

Note that, on the event $\{p_1(\xi_i) = 1\}$ we have $\eta_i = 1$. Therefore, using (3.21) and the fact that the r.v.'s ξ_0, \dots, ξ_{n-1} are i.i.d., we get

$$(3.22) \quad \mathbb{E}_k [W_n^{-a} | A_{i,n}] = \tilde{\mathbb{E}}_k [W_{n-i}^{-a}].$$

By the convexity of the function $x \mapsto x^{-a}$, we have $\sup_{n \geq i} \tilde{\mathbb{E}}_k [W_{n-i}^{-a}] \leq \tilde{\mathbb{E}}_k W^{-a}$ (see [13] Lemma 2.1). Thus, by (3.20) and (3.22), we obtain

$$(3.23) \quad \mathbb{E}_k [W^{-a}] \leq \tilde{\mathbb{E}}_k [W^{-a}].$$

Note that, conditioning by the events $\{p_1(\xi_0) = 1\}$ and $\{p_1(\xi_0) < 1\}$, we have

$$\mathbb{E} p_1^k(\xi_0) m_0^a = (1 - \eta) \tilde{\mathbb{E}} p_1^k(\xi_0) m_0^a + \eta,$$

with $\eta = \mathbb{P}(p_1(\xi_0) = 1)$. So the condition $\mathbb{E} p_1^k(\xi_0) m_0^a < 1$ implies that $\tilde{\mathbb{E}} p_1^k(\xi_0) m_0^a < 1$. Then, by (3.18) applied under the probability $\tilde{\mathbb{P}}$, and the fact that $\tilde{\mathbb{P}}(p_1(\xi_0) < 1) = 1$, we get $\tilde{\mathbb{E}}_k [W^{-a}] < \infty$. Therefore, by (3.23), it follows that

$$(3.24) \quad \mathbb{E} p_1^k(\xi_0) m_0^a < 1 \quad \text{implies} \quad \mathbb{E}_k W^{-a} < \infty,$$

which ends the proof of Theorem 2.1.

4. SMALL VALUE PROBABILITY IN THE NON-EXTINCTION CASE

In this section we prove Theorem 2.3. We start with the proof of part a). For $k \geq 1$ and $j \geq k$, define

$$(4.1) \quad a_{k,n}(j) = \frac{\mathbb{P}(Z_n = j | Z_0 = k)}{\gamma_k^n},$$

with $\gamma_k = \mathbb{P}_k(Z_1 = k)$. By the Markov property, we have

$$\mathbb{P}_k(Z_{n+1} = j) \geq \mathbb{P}_k(Z_1 = k) \mathbb{P}_k(Z_n = j).$$

Dividing by γ_k^{n+1} leads to

$$(4.2) \quad a_{k,n+1}(j) \geq a_{k,n}(j).$$

Therefore, by the monotone ratio theorem, we obtain

$$\lim_{n \rightarrow \infty} \uparrow a_{k,n}(j) = q_{k,j} \in \bar{\mathbb{R}}.$$

We shall prove that $q_{k,j}$ satisfies the properties claimed in the theorem. If j is such that $\mathbb{P}_k(Z_n = j) = 0$ for any $n \geq 0$, then $a_{k,n}(j) = 0$ for any $n \geq 0$, so that $\lim_{n \rightarrow \infty} a_{k,n}(j) = 0 = q_{k,j}$. If there exists $l \geq 0$ such that $\mathbb{P}_k(Z_l = j) > 0$, then $q_{k,j} \geq a_{k,l}(j) = \mathbb{P}_k(Z_l = j)/\gamma_k^l > 0$.

Now we show by induction that for all $j \geq k$, we have

$$H(j) : \sup_{n \in \mathbb{N}} a_{k,n}(j) = a_k(j) < \infty.$$

For $j = k$, we have $a_k(k) = 1$. Assume that $j \geq k + 1$ and that $H(i)$ is true for all $k \leq i \leq j - 1$. By the total probability formula, we obtain

$$\frac{\mathbb{P}_k(Z_{n+1} = j)}{\gamma_k^{n+1}} = \frac{1}{\gamma_k} \sum_{i=k}^j \mathbb{P}_k(Z_{n+1} = j | Z_n = i) \frac{\mathbb{P}_k(Z_n = i)}{\gamma_k^n},$$

which is equivalent to

$$(4.3) \quad a_{k,n+1}(j) = \frac{1}{\gamma_k} \left[\sum_{i=k}^{j-1} p(i, j) a_{k,n}(i) + \gamma_j a_{k,n}(j) \right]$$

with $p(i, j) = \mathbb{P}(Z_1 = j | Z_0 = i)$. Using the fact that $a_{k,n}(j) \leq a_{k,n+1}(j)$, we get by induction that

$$\sup_{n \in \mathbb{N}} a_{k,n+1}(j)(\gamma_k - \gamma_j) \leq \sum_{i=k}^{j-1} p(i, j) a_k(i) < \infty.$$

Thus $q_{k,j} < \infty$ for all $j \geq k + 1$ and $k \geq 1$. Furthermore, taking the limit as $n \rightarrow \infty$ in (4.3), leads to the following recurrent relation for $q_{k,j}$:

$$q_{k,k} = 1, \quad \gamma_k q_{k,j} = \sum_{i=k}^j p(i, j) q_{k,i} \quad (j \geq k + 1).$$

This ends the proof of part a) of Theorem 2.3.

Now we prove part b) of Theorem 2.3. We give a proof that the radius of convergence of the power series Q_k is equal to 1. The method is new even in the case of the Galton-Watson process. We start with a lemma.

Lemma 4.1. *Let $k \geq 1$. Assume that $\mathbb{P}(p_1(\xi_0) < 1) = 1$ and that there exists $\varepsilon > 0$ such that $\mathbb{E}[m_0^{r_k + \varepsilon}] < \infty$, where r_k is the solution of the equation $\gamma_k = \mathbb{E}m_0^{-r_k}$. Then, for any $r > r_k$, we have*

$$(4.4) \quad \lim_{n \rightarrow \infty} \uparrow \frac{\mathbb{E}_k Z_n^{-r}}{\gamma_k^n} < \infty.$$

Proof. By the Markov property,

$$\mathbb{E}_k [Z_{n+1}^{-r}] \geq \mathbb{E}_k [Z_{n+1}^{-r} | Z_1 = k] \mathbb{P}_k(Z_1 = k) = \gamma_k \mathbb{E}_k [Z_n^{-r}],$$

which proves that the sequence $(\mathbb{E}_k [Z_n^{-r}] / \gamma_k^n)_{n \in \mathbb{N}}$ is increasing. We show that it is bounded. For $n \geq 1$ and $m \geq 0$, we have the following well-known branching

property for Z_n :

$$(4.5) \quad Z_{n+m} = \sum_{i=1}^{Z_m} Z_{n,i}^{(m)},$$

where, under \mathbb{P}_ξ , the random variables $Z_{n,i}^{(m)}$ ($i \geq 1$) are i.i.d., independent of Z_m , whose conditional probability law satisfies $\mathbb{P}_\xi(Z_{n,i}^{(m)} \in \cdot) = \mathbb{P}_{T^m \xi}(Z_n \in \cdot)$, with T^m the shift operator defined by $T^m(\xi_0, \xi_1, \dots) = (\xi_m, \xi_{m+1}, \dots)$. Intuitively, relation (4.5) shows that, conditionally on $Z_m = i$, the annealed law of the process Z_{n+m} is the same as that of a new process Z_n starting with i individuals.

Using (4.5) with $m = 1$, the independence between Z_1 and $Z_{n,i}^{(1)}$ ($i \geq 1$) and the fact that $\mathbb{E}_i Z_n^{-r} \leq \mathbb{E}_{k+1} Z_n^{-r}$ for all $i \geq k+1$, we have

$$(4.6) \quad \begin{aligned} \mathbb{E}_k [Z_{n+1}^{-r}] &= \mathbb{E}_k [Z_{n+1}^{-r} | Z_1 = k] \mathbb{P}_k(Z_1 = k) \\ &\quad + \sum_{i=k+1}^{\infty} \mathbb{E} \left[\left(\sum_{h=1}^i Z_{n,h}^{(1)} \right)^{-r} \middle| Z_1 = i \right] \mathbb{P}_k(Z_1 = i) \\ &\leq \gamma_k \mathbb{E}_k [Z_n^{-r}] + \mathbb{E}_{k+1} [Z_n^{-r}]. \end{aligned}$$

We shall use the following change of measure: for $k \geq 1$ and $r > 0$, let $\mathbb{P}_k^{(r)}$ be a new probability measure determined by

$$(4.7) \quad \mathbb{E}_k^{(r)}[T] = \frac{\mathbb{E}_k [\Pi_n^{-r} T]}{c_r^n}$$

for any \mathcal{F}_n -measurable random variable T , where $c_r = \mathbb{E} m_0^{-r}$. By (4.7), we obtain

$$(4.8) \quad \mathbb{E}_{k+1} [Z_n^{-r}] = \mathbb{E}_{k+1}^{(r)} [W_n^{-r}] c_r^n,$$

with $\sup_{n \in \mathbb{N}} \mathbb{E}_{k+1}^{(r)} [W_n^{-r}] = \mathbb{E}_{k+1}^{(r)} [W^{-r}]$ (see [13], Lemma 2.1). Moreover, we have $\mathbb{E}^{(r)} [p_1^{k+1}(\xi_0) m_0^r] = \gamma_{k+1} / \mathbb{E} m_0^{-r} < 1$ for any $r < r_{k+1}$. So by Theorem 2.1 we get $\mathbb{E}_{k+1}^{(r)} [W^{-r}] = C(r) < \infty$ and then $\mathbb{E}_{k+1} [Z_n^{-r}] \leq C(r) c_r^n$ for any $r < r_k + \varepsilon < r_{k+1}$. Coming back to (4.6) with $r < r_k + \varepsilon$, we get by induction that

$$(4.9) \quad \mathbb{E}_k [Z_{n+1}^{-r}] \leq \gamma_k^{n+1} + C \sum_{j=0}^n \gamma_k^{n-j} c_r^j.$$

Choose $r > r_k$ such that $c_r < \gamma_k$. Then, we have, as $n \rightarrow \infty$,

$$(4.10) \quad \frac{\mathbb{E}_k [Z_{n+1}^{-r}]}{\gamma_k^{n+1}} \leq 1 + \frac{C}{\gamma_k} \sum_{j=0}^n \left(\frac{c_r}{\gamma_k} \right)^j \rightarrow \frac{C}{\gamma_k - c_r}.$$

Thus the sequence $(\mathbb{E}_k [Z_n^{-r}] / \gamma_k^n)_{n \in \mathbb{N}}$ is bounded and (4.4) holds for any $r \in (r_k, r_k + \varepsilon)$. Using the fact that $\mathbb{E}_k [Z_{n+1}^{-r'}] \leq \mathbb{E}_k [Z_{n+1}^{-r}]$ for any $r' > r$, the result follows for any $r > r_k$, which ends the proof of the lemma. \square

Remark 4.2. *From the results stated above, with some additional analysis one can obtain the equivalent of the harmonic moments $\mathbb{E}Z_n^{-r}$ for any $r > 0$. However, it is delicate to have an expression of the concerned constant in the equivalence. This will be considered in a forthcoming paper.*

Now we show that the radius of convergence R of the power series $Q_k(t) = \sum_{j=k}^{\infty} q_{k,j} t^j$ is equal to 1. Using the fact that $\sum_{j=k}^{\infty} \mathbb{P}_k(Z_n = j) = 1$, part a) of Theorem 2.3 and the monotone convergence theorem, we have

$$\lim_{n \rightarrow \infty} \uparrow \gamma_k^{-n} \sum_{j=k}^{\infty} \mathbb{P}_k(Z_n = j) = \sum_{j=k}^{\infty} q_{k,j} = +\infty,$$

which proves that $R \leq 1$. We prove that $R = 1$ by showing that $\sum_{j=k}^{+\infty} j^{-r} q_{k,j} < \infty$ for $r > 0$ large enough. Using part a) of Theorem 2.3, the monotone convergence theorem and Lemma 4.1, we have, for any $r > r_k$,

$$(4.11) \quad \sum_{j=k}^{+\infty} j^{-r} q_{k,j} = \sum_{j=k}^{+\infty} j^{-r} \lim_{n \rightarrow \infty} \uparrow \frac{\mathbb{P}_k(Z_n = j)}{\gamma_k^n} = \lim_{n \rightarrow \infty} \uparrow \frac{\mathbb{E}_k Z_n^{-r}}{\gamma_k^n} < \infty,$$

which proves part b).

Now we prove part c) of Theorem 2.3. Using part a), the definition of $G_{k,n}$ and the monotone convergence theorem, we get (2.16). To prove the functional relation (2.17), recall that $G_{k,1}(t) = \sum_{j=k}^{\infty} p(k, j) t^j = \mathbb{E} f_0^k(t)$. By (2.14) and Fubini's theorem, we get

$$\begin{aligned} \gamma_k Q_k(t) &= \sum_{j=k}^{\infty} \sum_{i=k}^{\infty} q_{k,i} p(i, j) \mathbf{1}(i \leq j) t^j \\ &= \sum_{i=k}^{\infty} q_{k,i} \sum_{j=i}^{\infty} p(i, j) t^j \\ &= \sum_{i=k}^{\infty} q_{k,i} \mathbb{E} [f_0^i(t)] \\ &= \mathbb{E} \left[\sum_{i=k}^{\infty} q_{k,i} f_0^i(t) \right] \\ &= \mathbb{E} [Q_k(f_0(t))]. \end{aligned}$$

This proves the functional relation (2.17).

We now prove that the previous functional relation characterizes the function Q_k . To this end it suffices to show the unicity of the solution of (2.17). Assume that there exists a power series $\hat{Q}(t) = \sum_{j=0}^{\infty} \hat{q}_{k,j} t^j$ on $[0, 1)$ which verifies (2.17) with the initial condition $q_{k,k} = \hat{q}_{k,k} = 1$. We first show by induction in n that $\hat{Q}^{(n)}(0) = 0$ for all $n \in \{0, \dots, k-1\}$. Since $f_0(0) = 0$ and $\gamma_k \in (0, 1)$, by (2.17), we get $\gamma_k \hat{Q}(0) = \hat{Q}(0)$, which implies that $\hat{Q}^{(0)}(0) = \hat{Q}(0) = 0$. By the induction hypothesis we have that $\hat{Q}^{(j)}(0) = 0$ for all $j \in \{0, \dots, n-1\}$ for some $n < k-1$. We show that $\hat{Q}^{(n)}(0) = 0$.

Using Faà di Bruno's formula, we have

$$(4.12) \quad (\hat{Q} \circ f_0)^{(n)}(t) = \sum_{j=1}^n \hat{Q}^{(j)}(f_0(t)) B_{n,j} \left(f_0^{(1)}(t), \dots, f_0^{(n-j+1)}(t) \right),$$

where $B_{n,j}$ are the Bell polynomials, defined for any $1 \leq j \leq n$ by

$$B_{n,j}(x_1, x_2, \dots, x_{n-j+1}) = \sum \frac{n!}{i_1! i_2! \dots i_{n-j+1}!} \left(\frac{x_1}{1!} \right)^{i_1} \left(\frac{x_2}{2!} \right)^{i_2} \dots \left(\frac{x_{n-j+1}}{(n-j+1)!} \right)^{i_{n-j+1}},$$

where the sum is taken over all sequences (i_1, \dots, i_{n-j+1}) of non-negative integers such that $i_1 + \dots + i_{n-j+1} = j$ and $i_1 + 2i_2 + \dots + (n-j+1)i_{n-j+1} = n$. In particular $B_{n,n}(x_1) = x_1^n$. Applying (4.12) and using the fact that $f_0(0) = 0$, $B_{n,n}(f_0^{(1)}(0)) = f_0^{(1)}(0)^n$ and $\hat{Q}^{(j)}(0) = 0$ for all $j \in \{0, \dots, n-1\}$, we get

$$(4.13) \quad (\hat{Q} \circ f)^{(n)}(0) = \hat{Q}^{(n)}(0) (f_0^{(1)}(0))^n.$$

Then taking the derivative of order n of both sides of (2.17) and using (4.13), we obtain that $\gamma_k \hat{Q}^{(n)}(0) = \hat{Q}^{(n)}(0) \gamma_n$ for $n < k-1$, which implies that $\hat{Q}^{(n)}(0) = 0$.

Now we show that $\hat{q}_{k,j} = q_{k,j}$ for any $j \geq k+1$. Using Fubini's theorem, the fact that f_0, \dots, f_{n-1} are i.i.d. and iterating (2.17), we get

$$(4.14) \quad \mathbb{E}[Q_k(\bar{g}_n(t))] = \gamma_k^n Q_k(t) \quad \text{and} \quad \mathbb{E}[\hat{Q}_k(\bar{g}_n(t))] = \gamma_k^n \hat{Q}_k(t),$$

where $\bar{g}_n(t) = f_{n-1} \circ \dots \circ f_0(t)$. By (4.14), for all $t \in [0, 1]$ and $n \in \mathbb{N}$, we have

$$(4.15) \quad \begin{aligned} |Q_k(t) - \hat{Q}_k(t)| &= \gamma_k^{-n} \left| \mathbb{E}[Q_k(\bar{g}_n(t)) - \hat{Q}_k(\bar{g}_n(t))] \right| \\ &= \gamma_k^{-n} \left| \sum_{j=k}^{\infty} (q_{k,j} - \hat{q}_{k,j}) \mathbb{E}[\bar{g}_n^j(t)] \right| \\ &\leq \gamma_k^{-n} \sum_{j=k+1}^{\infty} |q_{k,j} - \hat{q}_{k,j}| G_{j,n}(t), \end{aligned}$$

where $G_{j,n}(t)$ is the generating function of Z_n starting with j individuals. To conclude the proof of the unicity it is enough to show that

$$(4.16) \quad \lim_{n \rightarrow \infty} \sum_{j=k+1}^{\infty} |q_{k,j} - \hat{q}_{k,j}| \gamma_k^{-n} G_{j,n}(t) = 0.$$

We prove (4.16) using the Lebesgue dominated convergence theorem. Note that, by (2.16), for all $n \in \mathbb{N}$,

$$(4.17) \quad \gamma_j^{-n} G_{j,n}(t) \leq Q_j(t).$$

Therefore, using the fact that $\gamma_j < \gamma_k$ for all $j \geq k+1$, we have

$$\lim_{n \rightarrow \infty} \gamma_k^{-n} G_{j,n}(t) = \lim_{n \rightarrow \infty} \left(\frac{\gamma_j}{\gamma_k} \right)^n \gamma_j^{-n} G_{j,n}(t) \leq \lim_{n \rightarrow \infty} \left(\frac{\gamma_j}{\gamma_k} \right)^n Q_j(t) = 0,$$

and $\gamma_k^{-n} G_{j,n}(t) \leq Q_j(t)$. Now we show that $\sum_{j=k+1}^{\infty} |q_{k,j} - \hat{q}_{k,j}| Q_j(t) < \infty$ for all $t \in [0, 1)$. Indeed, by part b) of Theorem 2.3, we have $\sum_{j=k}^{\infty} q_{k,j} j^{-r} < \infty$ for any $r > r_k$. In particular for a fixed $r > r_k$, there exists a constant $C > 0$ such that, for all $j \geq 1$, $i \geq j$, it holds $q_{j,i} \leq C i^r$. Therefore,

$$Q_j(t) \leq C \sum_{i=j}^{\infty} i^r t^i \leq C t^j \sum_{i=0}^{\infty} (i+j)^r t^i \leq C \left(2^r t^j \sum_{i=0}^{\infty} i^r t^i + 2^r t^j \sum_{i=0}^{\infty} j^r t^i \right) \leq C_r j^r t^j.$$

Since Q_k and \hat{Q}_k are power series whose radii of convergence are equal to 1, we have, for any $t < 1$,

$$(4.18) \quad \sum_{j=k+1}^{\infty} q_{k,j} Q_j(t) \leq C_r \sum_{j=k+1}^{\infty} q_{k,j} j^r t^j < \infty, \quad \text{and} \quad \sum_{j=k+1}^{\infty} \hat{q}_{k,j} Q_j(t) < \infty.$$

Using the dominated convergence theorem, we see that

$$\lim_{n \rightarrow \infty} \gamma_k^{-n} \sum_{j=k+1}^{\infty} |q_{k,j} - \hat{q}_{k,j}| G_{j,n}(t) = 0.$$

Therefore, from (4.15) we conclude that $Q_k(t) = \hat{Q}_k(t)$ for all $t \in [0, 1)$. This ends the proof of Theorem 2.3.

REFERENCES

- [1] V. I. Afanasyev, C. Böinghoff, G. Kersting, and V. A. Vatutin. Limit theorems for weakly subcritical branching processes in random environment. *J. Theoret. Probab.*, 25(3):703–732, 2012.
- [2] V. I. Afanasyev, C. Böinghoff, G. Kersting, and V. A. Vatutin. Conditional limit theorems for intermediately subcritical branching processes in random environment. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(2):602–627, 2014.
- [3] K. B. Athreya and S. Karlin. Branching processes with random environments: II: Limit theorems. *Ann. Math. Stat.*, 42(6):1843–1858, 1971.
- [4] K. B. Athreya and S. Karlin. On branching processes with random environments: I: Extinction probabilities. *Ann. Math. Stat.*, 42(5):1499–1520, 1971.
- [5] K. B. Athreya and P. E. Ney. *Branching processes*, volume 28. Springer-Verlag Berlin, 1972.
- [6] V. Bansaye and J. Berestycki. Large deviations for branching processes in random environment. *Markov Process. Related Fields*, 15(4):493–524, 2009.
- [7] V. Bansaye and C. Böinghoff. Upper large deviations for branching processes in random environment with heavy tails. *Electron. J. Probab.*, 16(69):1900–1933, 2011.
- [8] V. Bansaye and C. Böinghoff. Lower large deviations for supercritical branching processes in random environment. *Proc. Steklov Inst. Math.*, 282(1):15–34, 2013.
- [9] V. Bansaye and C. Böinghoff. Small positive values for supercritical branching processes in random environment. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(3):770–805, 2014.
- [10] C. Böinghoff. Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions. *Stoch. Process. Appl.*, 124(11):3553–3577, 2014.
- [11] C. Böinghoff and G. Kersting. Upper large deviations of branching processes in a random environment - offspring distributions with geometrically bounded tails. *Stoch. Process. Appl.*, 120(10):2064–2077, 2010.

- [12] I. Grama, Q. Liu, and E. Miqueu. Berry-Esseen's bound and Cramér's large deviation expansion for a supercritical branching process in a random environment. *arXiv preprint arXiv:1602.02081*, 2016.
- [13] C. Huang and Q. Liu. Moments, moderate and large deviations for a branching process in a random environment. *Stoch. Process. Appl.*, 122(2):522–545, 2012.
- [14] M. V. Kozlov. On large deviations of branching processes in a random environment: geometric distribution of descendants. *Discrete Math. Appl.*, 16(2):155–174, 2006.
- [15] Q. Liu. Asymptotic properties of supercritical age-dependent branching processes and homogeneous branching random walks. *Stoch. Process. Appl.*, 82(1):61–87, 1999.
- [16] M. Nakashima. Lower deviations of branching processes in random environment with geometrical offspring distributions. *Stoch. Process. Appl.*, 123(9):3560–3587, 2013.
- [17] W. L. Smith and W. E. Wilkinson. On branching processes in random environments. *Ann. Math. Stat.*, 40(3):814–827, 1969.
- [18] D. Tanny. A necessary and sufficient condition for a branching process in a random environment to grow like the product of its means. *Stoch. Process. Appl.*, 28(1):123–139, 1988.
- [19] V. A. Vatutin. A refinement of limit theorems for the critical branching processes in random environment. In *Workshop on Branching Processes and their Applications. Lect. Notes Stat. Proc.*, volume 197, pages 3–19. Springer, Berlin, 2010.
- [20] V. A. Vatutin and X. Zheng. Subcritical branching processes in a random environment without the Cramer condition. *Stoch. Process. Appl.*, 122(7):2594–2609, 2012.

Current address, Grama, I.: Université de Bretagne-Sud, LMBA UMR CNRS 6205, Vannes, France

E-mail address: `ion.grama@univ-ubs.fr`

Current address, Liu, Q.: Université de Bretagne-Sud, LMBA UMR CNRS 6205, Vannes, France

E-mail address: `quansheng.liu@univ-ubs.fr`

Current address, Miqueu, E.: Université de Bretagne-Sud, LMBA UMR CNRS 6205, Vannes, France

E-mail address: `eric.miqueu@univ-ubs.fr`